SECONDARY FLOWS IN MAGNETOHYDRODYNAMIC FLOW AROUND A FINITE-WIDTH PLATE WITH INTERNAL ELECTROMAGNETIC FIELD SOURCES

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The results of theoretical and experimental investigations of magnetohydrodynamic flow around bodies in uniform external fields are well known. Problems involving magnetohydrodynamic flow around finite-dimension solids in electromagnetic fields produced by sources located inside or at the surface of the solid have been investigated to a much lesser extent. Such sources make it possible to control to a large extent the distribution of electromagnetic volumetric forces (EVF) in the liquid and modify both the hydrodynamic flow pattern and the hydrodynamic resistance, which is evident from the numerical investigations [1-3] performed for medium Reynolds numbers. The present paper is concerned with an analytical investigation of secondary flows that arise in the flow around a finite-width flat plate with an internal source [4] for large Reynolds number and small values of the magnetohydrodynamic interaction parameter N. The exact solution has been obtained for the first term of an expansion of the three-dimensional flow field under consideration with respect to the N parameter. The obtained solution is used for investigating the qualitative characteristics of the wake stream formed behind the plate and the conditions under which the secondary flow is close to a two-dimensional flow.

1. For the sake of convenience in presentation, we shall briefly recall the basic elements of the system under consideration [4], which consists of a zero-thickness plate of infinite length along the y coordinate, whose width is equal to a along the x axis. The plate is immersed in the flow of an incompressible, conducting liquid having the velocity $u_0 e_x$ at infinity upstream along the flow. The internal, conduction-type source consists of a magnetic system in the form of a set of periodic current cells (surface cells, located in the plane of the plate),

$$\mathbf{i}(x, y) = J_0\left[i(x)\,\mathbf{e}_x + \frac{i}{k_0}\frac{di}{dx}\,\mathbf{e}_y\right]e^{ik_0y}, \quad i(x) = \begin{cases} i_0(x) & \text{for} \quad |x| \leqslant 1/2, \\ 0 & \text{for} \quad |x| > 1/2 \end{cases} \qquad (1.1)$$
$$(|i_0(x)|_{max} = 1)$$

and ideally sectional electrodes with a potential depending periodically on y ($\sim e^{ik_0y}$). It is evident from (1.1) that the magnetic system is assigned by a single dimensionless function $i_0(x)$, which is assumed to be real, and by the dimensionless wave number k_0 (it is rendered dimensionless by using a as the length scale). The form of the function $i_0(x)$ and the value of k_0 determine the character of the current cells (shown schematically in Fig. 2 in [4]) that constitute the magnetic system of the source, so that they depend to a large extent on the purpose of the magnetohydrodynamic device. If, for instance, the device is to be used for accelerating the oncoming flow or transforming the energy of the oncoming flow into electric energy, its magnetic system must consist of current cells extended along the x axis (i.e., along the flow); the currents will then basically be longitudinal, while the transverse, closing currents must be located only near the plate edges within a strip of definite width. A magnetic system of this type corresponds to the function $i_0(x)$ which is equal to unity over a larger part of the interval $|x| \leq 0.5$, decreasing to zero at the ends of this interval.

The secondary flows produced by electromagnetic forces are investigated specifically for the function

$$i_0(x) = \frac{\left(1 - e^{-(0.5+x)k_0}\right)\left(1 - e^{-(0.5-x)k_0}\right)}{\left(1 - e^{-0.5k_0}\right)^2}, \qquad (1.2)$$

which, for $k_0 \gg 1$, provides a magnetic system for flow acceleration that is close to the optimum one, as was demonstrated in [4].

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2. The vortex flow around the plate caused by electromagnetic forces is investigated here for the most interesting case of large Reynolds numbers. In this case, significant perturbation of the velocity field of the oncoming uniform flow occurs near the plate at a distance from it not exceeding $\lambda = (2\pi/k_0)a$ (along the normal). Assuming that the boundary layer thickness is small and that the force field and the region of perturbations of the velocity field are located far beyond the boundaries of the viscous boundary layer, we can investigate the velocity field outside this layer without considering the effect of viscosity. The dimensionless velocity field here satisfies the equations of hydrodynamics,

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla p + N\mathbf{f},\ \operatorname{div}\mathbf{V} = 0 \tag{2.1}$$

with the force field

$$\mathbf{f} = [\mathbf{E}\times\mathbf{H}] + \mathbf{H}(\mathbf{V}\cdot\mathbf{H}) - \mathbf{V}H^2, \tag{2.2}$$

the condition of absence of perturbations at infinity upstream along the flow, i.e.,

$$\mathbf{V}|_{x=-\infty} = \mathbf{e}_x, \tag{2.3}$$

and the condition for the absence of seepage through the plate.

In the induction-free approximation, the magnetic field in the liquid appearing in (2.2) does not depend on the velocity field and is determined by the surface currents (1.1) assigned in the plane of the plate.

Perturbations of the velocity field affect the electric field due to the development of space charges in the perturbed flow region, and this effect must not be neglected. The dimensionless potential of the electric field is described by the nonhomogeneous equation

$$\Delta\varphi(x,\ y,\ z) = \mathbf{H}\cdot\operatorname{curl}\mathbf{V}. \tag{2.4}$$

The quantities $H_0 = 2\pi J_0/c$, u_0H_0/c, and au_0H_0/c are used as the scales in rendering dimensionless **H**, **E**, and φ, respectively.

3. Under the assumption that $N \ll 1$, we seek the solution of Eqs. (2.1) and (2.4) in the following form:

$$\begin{aligned} \mathbf{V}(x,\ y,\ z) &= \mathbf{e}_x + N\mathbf{V}_1(x,\ y,\ z) + N^2\mathbf{V}_2(x,\ y,\ z) + \ldots, \\ p(x,\ y,\ z) &= Np_1(x,\ y,\ z) + N^2p_2(x,\ y,\ z) + \ldots, \\ \varphi(x,\ y,\ z) &= \varphi_0(x,\ y,\ z) + N\varphi_1(x,\ y,\ z) + N^2\varphi_2(x,\ y,\ z) + \ldots, \end{aligned} \tag{3.1}$$

while, in the exact formulation, we shall seek the solution only for the first-order velocity field. On the basis of Eqs. (2.1) and condition (2.3), the problem is reduced to the following one:

$$\partial\mathbf{V}_1/\partial x = -\nabla p_1 + \mathbf{f}_0,\ \operatorname{div}\mathbf{V}_1 = 0,\ \ \mathbf{f}_0 = [\mathbf{E}_0\times\mathbf{H}] + H_x\mathbf{H} - H^2\mathbf{e}_x, \tag{3.2}$$

$$\begin{aligned} \mathbf{E}_0 &= -\nabla\varphi_0(x,\ y,\ z); \\ \mathbf{V}_1|_{x=-\infty} &= 0. \end{aligned} \tag{3.3}$$

The function $\varphi_0(x,\ y,\ z)$ appearing in $\mathbf{f}_0$ is defined as the solution of the problem

$$\Delta\varphi_0(x, y, z) = 0,\ \ \frac{\partial\varphi_0}{\partial z}\bigg|_{z=\pm 0} = \pm\varkappa e^{ik_0y}i(x),\ \ \varphi_0|_\infty = 0 \tag{3.4}$$

for the boundary condition derived in [4] from the optimization condition.* The field **H** is determined by the assigned surface currents (1.1), which can be represented in the form of volume currents by means of the delta function $\delta(z)$,

$$\frac{J_0}{a}\mathbf{j}(x, y, z) = \frac{J_0}{a}e^{ik_0y}\left[i(x)\,\mathbf{e}_x + \frac{i}{k_0}\frac{di}{dx}\mathbf{e}_y\right]\delta(z) \tag{3.5}$$

and the problem of determining the dimensionless field **H** throughout the entire space is reduced to the following one:

*The constant $1 + 1/\gamma$ appears instead of $\varkappa$ in [4].

$$\operatorname{rot}\mathbf{H} = 2\mathbf{j}(x, y, z), \quad \operatorname{div}\mathbf{H} = 0, \quad \mathbf{H}|_\infty = 0. \tag{3.6}$$

The solutions of problems (3.4) and (3.6) are given in [4] in Fourier representation with respect to the x variable. In order to solve Eqs. (3.2), the force field $\mathbf{f}_0$ and, consequently, also the fields $\mathbf{E}_0$ and $\mathbf{H}$ must be in the Fourier representation with respect to the x and z variables. By representing the dimensionless current density (3.5) in the form of a two-dimensional expansion,

$$\mathbf{j}(x, y, z) = e^{ik_0 y}\iint\limits_{-\infty}^{\infty} \mathbf{j}(k_x, k_z)\, e^{i(k_x x + k_z z)}\, dk_x dk_z,$$

$$\mathbf{j}(k_x, k_z) = \frac{1}{2\pi}\left(\mathbf{e}_x - \frac{k_x}{k_0}\mathbf{e}_y\right) i(k_x), \quad i(k_x) = \frac{1}{2\pi}\int\limits_{-1/2}^{1/2} i_0(x)\, e^{-ik_x x}\, dx,$$

we obtain the solution of problem (3.6) in a similar form with the Fourier transform:

$$\mathbf{H}(k_x, k_z) = \frac{i}{\pi k_0}\, \frac{k_z(k_x\mathbf{e}_x + k_0\mathbf{e}_y) - (k_0^2 + k_x^2)\mathbf{e}_z}{k_x^2 + k_0^2 + k_z^2}\, i(k_x).$$

It should be noted that, due to the realness of the $i_0(x)$ function from (1.1), its Fourier transform figuring in (3.7) has the following property:

$$i(-k_x) = i^*(k_x). \tag{3.7}$$

The solution of problem (3.4) is given by

$$\varphi_0(x, y, z) = e^{ik_0 y}\iint\limits_{-\infty}^{\infty} \varphi(k_x, k_z)\, e^{i(k_x x + k_z z)}\, dk_x dk_z,$$
$$\varphi(k_x, k_z) = \frac{\varkappa}{\pi}\, \frac{i(k_x)}{k_x^2 + k_0^2 + k_z^2}. \tag{3.8}$$

After performing integration with respect to the variable k_z, integral (3.8) and the similar integral for $\mathbf{H}$ can readily be reduced to the corresponding expressions from [4], and the validity of the solutions obtained can thus be verified.

In reducing the force field $\mathbf{f}_0$ to the required form, it should be noted that, if the two real functions B_1 and B_2 have Fourier representations,

$$B_{1,2}(x, y, z) = \operatorname{Real} e^{ik_0 y}\iint\limits_{-\infty}^{\infty} B_{1,2}(k_x, k_z)\, e^{i(k_x x + k_z z)} dk_x dk_z,$$

then the product $C(x, y, z) = B_1(x, y, z)B_2(x, y, z)$ can be reduced to the following form:

$$C(x, y, z) = \operatorname{Real}\left\{\iint\limits_{-\infty}^{\infty} C^{(1)}(k_1, k_2)\, e^{i(k_1 x + k_2 z)} dk_1 dk_2 + e^{2ik_0 y}\iint\limits_{-\infty}^{\infty} C^{(2)}(k_1, k_2)\, e^{i(k_1 x + k_2 z)} dk_1 dk_2\right\},$$

$$C^{(1)}(k_1, k_2) = \frac{1}{8}\iint\limits_{-\infty}^{\infty} B_1\left(\frac{k_1 + q_1}{2}, \frac{k_2 + q_2}{2}\right) B_2^*\left(\frac{q_1 - k_1}{2}, \frac{q_2 - k_2}{2}\right) dq_1 dq_2, \tag{3.9}$$

$$C^{(2)}(k_1, k_2) = \frac{1}{8}\iint\limits_{-\infty}^{\infty} B_1\left(\frac{k_1 + q_1}{2}, \frac{k_2 + q_2}{2}\right) B_2\left(\frac{k_1 - q_1}{2}, \frac{k_2 - q_2}{2}\right) dq_1 dq_2.$$

It is evident from (3.9) that the Fourier components of the vector $\mathbf{f}_0$ (3.2) are expressed in terms of double integrals with respect to q_1 and q_2. Integration with respect to the variable q_2 can be performed in explicit form (by using residues), and, with an allowance for (3.7), the final result is reduced to the following:

$$\mathbf{f}_0 = \operatorname{Real}\left\{\iint\limits_{-\infty}^{\infty} \mathbf{f}^{(1)}(k_1, k_2)\, e^{i(k_1 x + k_2 z)} dk_1 dk_2 + e^{2ik_0 y}\iint\limits_{-\infty}^{\infty} \mathbf{f}^{(2)}(k_1, k_2)\, e^{i(k_1 x + k_2 z)} dk_1 dk_2\right\}; \tag{3.10}$$

$$\mathbf{f}^{(1,2)}(k_1, k_2) = \frac{\alpha^{(1,2)}}{4\pi} \int_{-\infty}^{\infty} \frac{i\left(\frac{k_1+q_1}{2}\right) i\left(\frac{k_1-q_1}{2}\right)}{\overset{+}{V}\overset{-}{V}\left[k_2^2 + \nu_2^2(k_1, q_1)\right]} \left\{\left[\frac{\varkappa - 1}{2}\left(\alpha^{(1,2)}\overset{+}{V}\overset{-}{V} - \right.\right.\right.$$

$$\left. - 4k_0^2 - k_1^2 - q_1^2\right) + \left(k_1^2\delta_q + q_1^2\delta_k\right)\Big](\overset{+}{V} + \overset{-}{V})\,\mathbf{e}_x + \gamma^{(1,2)}\frac{k_1}{4k_0}\left[2\varkappa\left(\nu_1^2 - \nu_2^2\right) + \overset{+}{V}\overset{-}{V}\right](\overset{+}{V} + \overset{-}{V})\,\mathbf{e}_y \tag{3.11}$$

$$\left. + k_2\left[k_1\left(\delta_q - \beta^{(1,2)}\varkappa\right)(\overset{+}{V} + \overset{-}{V}) + q_1\left(\delta_k - \gamma^{(1,2)}\varkappa\right)(\overset{+}{V} - \overset{-}{V})\right]\mathbf{e}_z\right\} dq_1.$$

Here and subsequently,

$$\overset{+}{V} = \sqrt{4k_0^2 + (k_1+q_1)^2}, \quad \overset{-}{V} = \sqrt{4k_0^2 + (k_1-q_1)^2}, \quad \nu_1 = \sqrt{4k_0^2 + k_1^2},$$

$$\nu_2 = \sqrt{\frac{4k_0^2 + k_1^2 + q_1^2 + \overset{+}{V}\overset{-}{V}}{2}}, \quad \delta_q = \frac{4k_0^2 + k_1^2 - q_1^2}{8k_0^2}, \quad \delta_k = \frac{4k_0^2 + q_1^2 - k_1^2}{8k_0^2}.$$

The constants α, β, and γ, have the values $\alpha^{(1)} = -1$, $\beta^{(1)} = 1$, $\gamma^{(1)} = 0$, $\alpha^{(2)} = 1$, $\beta^{(2)} = 0$, and $\gamma^{(2)} = 1$. It is evident from the above expressions that the force field $\mathbf{f}_0$ consists of two parts. One of them is volumetric in character and periodic with respect to y with the period $\lambda/2$. The other one is independent of y and does not contain a y-component, i.e., it represents a two-dimensional (x, z) force field.

4. We now turn to problem (3.2), (3.3). As a consequence of (3.10), its solution is also composed of similar two parts,

$$\mathbf{V}_1(x, y, z) = \mathbf{V}_c(x, z) + e^{2ik_0y}\tilde{\mathbf{V}}(x, z), \quad p(x, y, z) = p_c(x, z) + e^{2ik_0y}\tilde{p}(x, z); \tag{4.1}$$

$$\frac{\partial \mathbf{V}_c}{\partial x} = -\nabla p_c + \mathbf{f}_c(x, z), \quad \frac{\partial V_{cx}}{\partial x} + \frac{\partial V_{cz}}{\partial z} = 0; \tag{4.2}$$

$$\frac{\partial \tilde{\mathbf{V}}}{\partial x} = -\nabla\tilde{p} - 2ik_0\tilde{p}\mathbf{e}_y + \tilde{\mathbf{f}}(x, z), \quad \frac{\partial \tilde{V}_x}{\partial x} + \frac{\partial \tilde{V}_z}{\partial z} + 2ik_0\tilde{V}_y = 0; \tag{4.3}$$

$$\mathbf{V}_c|_{x=-\infty} = \tilde{\mathbf{V}}|_{x=-\infty} = 0; \quad p_c|_{x=-\infty} = \tilde{p}|_{x=-\infty} = 0. \tag{4.4}$$

Here $\mathbf{f}_c(x, z) = \iint_{-\infty}^{\infty} \mathbf{f}^{(1)}(k_1, k_2)\, e^{i(k_1x + k_2z)} dk_1 dk_2$, $\tilde{\mathbf{f}}(x, z) = \iint_{-\infty}^{\infty} \mathbf{f}^{(2)}(k_1, k_2)\, e^{i(k_1x + k_2z)} dk_1 dk_2$.

The particular solutions (marked by asterisks) of the nonhomogeneous equations (4.2) and (4.3) are sought in the form

$$\mathbf{V}_c^*(x, z) = \iint_{-\infty}^{\infty} \mathbf{V}_c^*(k_1, k_2)\, e^{i(k_1x + k_2z)} dk_1 dk_2 \tag{4.5}$$

(the calculations are performed in a similar manner for the other unknowns). From (4.2) and (4.3), we obtain

$$\mathbf{V}_c^*(k_1, k_2) = \frac{(\mathbf{k} - 2k_0\mathbf{e}_y)\left[(\mathbf{k} - 2k_0\mathbf{e}_y)\cdot\mathbf{f}^{(1)}(k_1, k_2)\right] - \left(k_1^2 + k_2^2\right)\mathbf{f}^{(1)}(k_1, k_2)}{ik_1\left(k_1^2 + k_2^2\right)},$$
$$\tilde{\mathbf{V}}(k_1, k_2) = \frac{k^2\mathbf{f}^{(2)}(k_1, k_2) - \left[\mathbf{k}\cdot\mathbf{f}^{(2)}(k_1, k_2)\right]\mathbf{k}}{ik_1k^2} \quad (\mathbf{k} = k_1\mathbf{e}_x + 2k_0\mathbf{e}_y + k_2\mathbf{e}_z). \tag{4.6}$$

It follows from (4.5), (4.6), and (3.11) that the sought solutions are represented by three-dimensional integrals in k_1, k_2, q_1 space. By using residues, we perform integration with respect to k_2 in explicit form and reduce all the sought quantities to two-dimensional integrals. In the general case of an arbitrary real function $i_0(x)$, it is difficult to use the integrals obtained for analyzing the flow in question, and they are therefore not reproduced here. There is an important particular case, the case of even functions $i_0(x)$, for which the Fourier transform $i(k_x)$ is a real even function, due to which the product $i[(k_1 - q_1)/2]i[(k_1 + q_1)/2]$, and also the functions $F(k_1, q_1)$, and $\Lambda(k_1, q_1)$ appearing in the integrals under consideration become even functions with respect to both variables k_1 and q_1, which allows us to reduce the sought expressions to the following real integrals

$$V_{cx}^{*}=-\int_0^\infty\int_0^\infty F(k_1,q_1)\frac{k_1 e^{-k_1 z}-\nu_2 e^{-\nu_2 z}}{\nu_2^2-k_1^2}\,\frac{\sin k_1 x}{k_1}\,dk_1 dq_1, \tag{4.7}$$

$$V_{cz}^{*}=-\int_0^\infty\int_0^\infty F(k_1,q_1)\frac{e^{-k_1 z}-e^{-\nu_2 z}}{\nu_2^2-k_1^2}\cos k_1 x\,dk_1 dq_1,$$

$$F(k_1,q_1)=\frac{i\left(\frac{k_1-q_1}{2}\right)i\left(\frac{k_1+q_1}{2}\right)}{\overset{+}{V}\overset{-}{V}}\left\{\left[(\varkappa-1)\left(4k_0^2+q_1^2\right)-\left(k_1^2+q_1^2\delta_k\right)\right](\overset{+}{V}+\overset{-}{V})+k_1 q_1(\delta_k-\varkappa+1)(\overset{+}{V}-\overset{-}{V})\right\};$$

$$\begin{aligned}\widetilde{V}_x^{*}=\int_0^\infty\int_0^\infty \Lambda(k_1,q_1)\frac{\sin k_1 x}{k_1}\Bigg\{\Bigg[&k_1^2(1+\delta_k-\varkappa)(\overset{+}{V}+\overset{-}{V})\\ &+k_1\nu_1^2(\varkappa-\delta_k)\frac{\overset{+}{V}-\overset{-}{V}}{q_1}\Bigg]\frac{e^{-\nu_1 z}}{\nu_1}+\Bigg[\left(k_1^2(\varkappa-1)-k_1^2-q_1^2\delta_k\right)(\overset{+}{V}+\overset{-}{V})+\\ &+\left(-\nu_1^2(\varkappa-1)+q_1^2\delta_k\right)k_1\frac{\overset{+}{V}-\overset{-}{V}}{q_1}\Bigg]\frac{e^{-\nu_2 z}}{\nu_2}\Bigg\}dk_1 dq_1,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\widetilde{V}_y^{*}=\frac{1}{2ik_0}\int_0^\infty\int_0^\infty \Lambda(k_1,q_1)\cos k_1 x\Bigg\{4k_0^2\Bigg[&(\delta_k-\varkappa+1)(\overset{+}{V}+\overset{-}{V})+\\ +(\varkappa-\delta_k)\frac{\overset{+}{V}-\overset{-}{V}}{k_1 q_1}\nu_1^2\Bigg]\frac{e^{-\nu_1 z}}{\nu_1}+\Bigg[&\left(-4k_0^2+(\varkappa-1)\left(q_1^2-k_1^2\right)\right)(\overset{+}{V}+\overset{-}{V})+(\varkappa-1)\left(\nu_1^2-q_1^2\right)k_1\frac{\overset{+}{V}-\overset{-}{V}}{q_1}\Bigg]\frac{e^{-\nu_2 z}}{\nu_2}\Bigg\}dk_1 dq_1,\end{aligned}$$

$$\widetilde{V}_z^{*}=\int_0^\infty\int_0^\infty \Lambda(k_1,q_1)\cos k_1 x\left\{\left(e^{-\nu_2 z}-e^{-\nu_1 z}\right)\left[(\varkappa-1-\delta_k)(\overset{+}{V}+\overset{-}{V})-\nu_1^2(\varkappa-\delta_k)\frac{\overset{+}{V}-\overset{-}{V}}{q_1 k_1}\right]\right\}dk_1 dq_1,$$

$$\Lambda(k_1,q_1)=\frac{i\left(\frac{k_1-q_1}{2}\right)i\left(\frac{k_1+q_1}{2}\right)}{\overset{+}{V}\overset{-}{V}}\,\frac{q_1^2}{\nu_1^2-\nu_2^2}.$$

The above solutions indicate that the transverse field components $\mathbf{V}_c^*$ and $\widetilde{\mathbf{V}}^*$ are even functions of x, while they all tend to zero for $x \to \pm\infty$. The longitudinal components V_{cx}^* and $\widetilde{V}_x^*$ are odd with respect to x and are nonvanishing at infinity upstream and downstream along the flow. Actually, for $x \to \infty$, the inner integrals (with respect to k_1) in expressions (4.7) and (4.8) tend to quantities equal to the products between $\pi/2$ and the values of the corresponding integrands for $k_1 = 0$ and $x = 1$, so that the sought limits assume the values

$$V_{cx}^{*}\big|_{x=\pm\infty}=\pm V_0(z),\quad V_0(z)=2\pi\int_0^\infty |i(q)|^2\left[(\varkappa-1)-\frac{q^2}{2k_0^2}\right]e^{-2\sqrt{k_0^2+q^2}\,z}\,dq, \tag{4.9}$$

$$\widetilde{V}_x^{*}\big|_{x=\pm\infty}=\pm \widetilde{V}_0(z),\quad \widetilde{V}_0(z)=\frac{\pi}{k_0^2}\int_0^\infty |i(q)|^2 q^2 e^{-2\sqrt{k_0^2+q^2}\,z}\,dq$$

(evidently, $\widetilde{V}_0(z)$ does not contain $\varkappa$ and coincides with $V_0(z)|_{\varkappa=1}$). Thus, the obtained particular solution of the nonhomogeneous equations (4.2) and (4.3) does not satisfy condition (4.4) for the absence of perturbations at infinity upstream along the flow and must be supplemented by suitable solutions of homogeneous equations. We finally have

$$\mathbf{V}_c=\mathbf{V}_c^{*}(x,z)+V_0(z)\,\mathbf{e}_x,\quad \widetilde{\mathbf{V}}=\widetilde{\mathbf{V}}^{*}(x,z)+\widetilde{V}_0(z)\,\mathbf{e}_x. \tag{4.10}$$

These relationships, together with (4.7)-(4.9), constitute the sought solution (4.1) of the first approximation for the velocity fields, which holds for an arbitrary even function $i_0(x)$.

5. The character of the flow in question is determined by the type of the internal source, i.e., the function $i_0(x)$ and the parameters k_0 and $\varkappa$. The simplest secondary flow

occurs in the wake behind the plate at a sufficiently large distance from it. The vectors $\mathbf{V}_c$ and $\tilde{\mathbf{V}}$ have in this case only longitudinal components, so that the wake flow constitutes a non-diffusing (due to the absence of viscosity) stream.

On the basis of (4.1), (4.9), and (4.10), the velocity distribution in the symmetry plane of the stream is given by

$$\mathbf{V}_{z=0} = 2N[V_0(0) + \tilde{V}_0(0)\cos 2k_0 y]\mathbf{e}_x, \tag{5.1}$$

i.e., it is determined by $V_0(z)|_{z=0} = V_0(0)$, and $\tilde{V}_0(z)|_{z=0} = \tilde{V}_0(0)$, for which, using relationships (4.9) (for z = 0) and the Parseval equations

$$\int_0^\infty |i(q)|^2 dq = \frac{1}{2\pi}\int_0^{1/2} i_0^2(x)\,dx, \quad \int_0^\infty |i(q)|^2 q^2 dq = \frac{1}{2\pi}\int_0^{1/2}\left(\frac{di_0}{dx}\right)^2 dx$$

we obtain the expressions

$$V_0(0) = \frac{(\varkappa-1)\left[1+4e^{-k_0}+e^{-2k_0}-\frac{3}{k_0}\left(1-e^{-2k_0}\right)\right]-\frac{1}{2k_0}\left(1-e^{-2k_0}\right)+e^{-k_0}}{2\left(1-e^{-0,5k_0}\right)^4},$$

$$\tilde{V}_0(0) = -V_0(0)|_{\varkappa-1=0}.$$

The dependence of $V_0(0)$ on k_0 shown in Fig. 1 is given for $\varkappa$ = 1.0, 1.2, 1.5, and 2.0 (curves 1-4, respectively), while curve 1 also describes $-\tilde{V}_0(0)$. Generally speaking, curves 1 and 4 contain complete information on the quantity in question because of the linear dependence of $V_0(0)$ on $\varkappa - 1$. From this point of view, curves 2 and 3 are "redundant," and are given here only for the sake of visual clarity. It is evident from the diagrams that, for $k_0 < k_0^*(\varkappa)$ the mean velocity in the symmetry plane of the stream is negative, while $k_0^*(\varkappa)$ is approximately equal to 0.7π for $\varkappa = 2$, increasing to 1.7π as $\varkappa$ decreases to 1.2.

For small values $k_0 \sim \pi$, the investigated velocity field has another characteristic feature: The values of $V_0(0)$, and $\tilde{V}_0(0)$ in (5.1) have the same order of magnitude, while, for large values of k_0 (and finite values of $\varkappa - 1$), the ratio of the scale of the "variable" (i.e., y-dependent) velocity component in the stream to the scale of the mean velocity is equal to

$$\frac{\tilde{V}_0(0)}{V_0(0)} = O\left[\frac{1}{k_0(\varkappa-1)}\right]. \tag{5.2}$$

Thus, for large values of k_0, the stream produced by electromagnetic forces is two-dimensional, while, for $k_0 \sim \pi$, it consists of "tongues" periodic with respect to y.

The velocity profile in the stream is similar to the velocity profile at the middle of the plate, regardless of the characteristics of the electromagnetic field source, since, according to (4.9) and (4.10),

$$V_{cx}|_{x=\infty} = 2V_{cx}|_{x=0} = 2V_0(z),\ \tilde{V}_x|_{x=\infty} = 2\tilde{V}_x|_{x=0} = 2\tilde{V}_0(z).$$

One can visualize the profile shape by referring to Fig. 2, which shows the $V_0(z)$ functions calculated for (1.2) for three characteristic values of k_0 (it should be noted that the product $k_0 z$ is laid off on the vertical axis in this case). Since the dependence of $V_0(z)$ on the parameter $\varkappa - 1$ is linear, the curves are given only for two values of $\varkappa$ = 1.5 (solid curves) and $\varkappa$ = 1.0 (dashed curves), while these values also characterize the function $\tilde{V}_0(z)$, since $\tilde{V}_0(z) = -V_0(z)|_{\varkappa=1}$. It is evident that the shape of the $V_0(z)$ profile for small k_0 values is different from the corresponding shape for large k_0 values. While the profile is monotonic for $k_0 \gg 1$,

$$V_0(z) = V_0(0)e^{-2k_0 z},$$

it becomes nonmonotonic with a reduction in k_0. It is evident from Fig. 2 that the velocity in the symmetry plane of the stream is negative for $k_0 = \pi$, and $\varkappa$ = 1.5, it has a maximum positive value at a distance of the order of $1/5k_0 = \lambda/10\pi$ from the symmetry plane, and it drops to zero at a distance of $\sim\lambda/4$, as in the case of large k_0 values.

If we have an idea of the character of the flow near the plate that is described by the velocity field $\mathbf{V}_c$, we can compose the velocity profiles at the middle of the plate and the

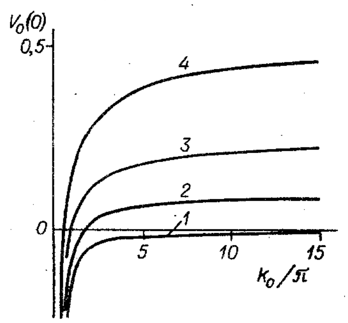

Fig. 1

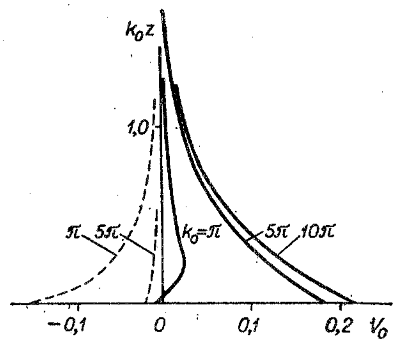

Fig. 2

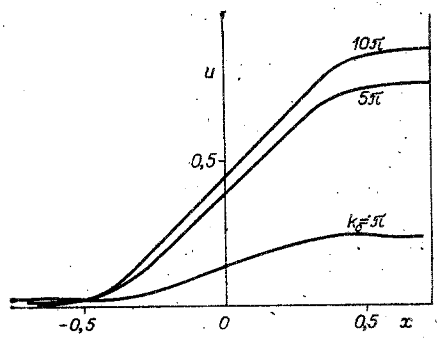

Fig. 3

velocity distribution $V_{cx}|_{z=0} = u$ along the plane of the plate. The former are shown in Fig. 2, and the latter in Fig. 3 for $\varkappa = 2$ and three different values of k_0. It is evident that, for all $k_0 \geq \pi$, the longitudinal velocity increase along the plate surface has the same character as the process whereby the velocity reaches the $2V_0(0)$ value; in particular, the velocity u practically coincides with its limiting value even at the 1/10 distance from the trailing edge.

The three-dimensional addition to the velocity field for $k_0 \gg 1$ is characterized by the estimate (5.2); it is difficult to obtain such estimates for the transverse components, since $\tilde{V}_y$ and $\tilde{V}_z$ are expressed by complex two-dimensional integrals (4.8). However, calculations show that, for all $k_0 \gg \pi$, the scales of $\tilde{V}_y$ and $\tilde{V}_z$ are considerably smaller than the scale of $\tilde{V}_x$ and that at least the inequalities

$$|\tilde{V}_y| < (1/k_0)|\tilde{V}_x|, \ |\tilde{V}_z| < (1/k_0)|\tilde{V}_x| \tag{5.3}$$

hold for the scales of the corresponding quantities.

6. In conclusion, we turn to the equations describing the second term of expansion (3.1):

$$\partial \mathbf{V}_2/\partial x = -(\mathbf{V}_1 \cdot \nabla)\mathbf{V}_1 - \nabla p_2 + \mathbf{f}_1, \ \operatorname{div} \mathbf{V}_2 = 0, \tag{6.1}$$

where $\mathbf{f}_1 = [\mathbf{E}_1 \times \mathbf{H}] + (\mathbf{V}_1 \cdot \mathbf{H})\mathbf{H} - \mathbf{H}^2\mathbf{V}_1$ is determined by the distribution of $\mathbf{H}$ and the solutions $\mathbf{V}_1$ and $\mathbf{E}_1$ of the first approximation. As in the case of $\mathbf{V}_1$, the solution $\mathbf{V}_2$ is composed of the two-dimensional vector field $\mathbf{V}_{2c}$, which is independent of the coordinate y, and an addition periodic with respect to y, which vanishes after averaging with respect to this coordinate.

The excessive cumbersomeness of the expressions for $\mathbf{V}_1$ and $\mathbf{f}_1$ makes it impossible to obtain the exact solution of these equations, and this makes it necessary to use numerical methods. However, in using numerical methods, it is naturally desirable to eliminate the constraint $N \ll 1$ and investigate the problem stated in a more general way. However, the three-dimensionality of the flow in question, characterized by large gradients, presents here an almost insuperable obstacle in the general case, and the question arises whether two-dimensional idealization can be used for obtaining quantities averaged with respect to y.

A partial answer to this question is obtained on the basis of Eq. (6.1) whence it follows that, for $k_0 >> 1$, the effect of the $\tilde{\mathbf{V}}e^{2ik_0y}$ component of the first-approximation solution on sought solution $\mathbf{V}_{2c}$ is slight, so that two-dimensional idealization describes the required solution at least with an accuracy to $O(N^2)$. Actually, with an allowance for (4.1), the component of $(\mathbf{V}_1\circ\nabla)\mathbf{V}_i$ from the right-hand side of (6.1) that is independent of y has the form

$$[(\mathbf{V}_1\cdot\nabla)\mathbf{V}_1]_c = \left(V_{cx}\frac{\partial V_{cx}}{\partial x} + V_{cz}\frac{\partial V_{cx}}{\partial z}\right)\mathbf{e}_x + \left(V_{cx}\frac{\partial V_{cz}}{\partial x} + V_{cz}\frac{\partial V_{cz}}{\partial z}\right)\mathbf{e}_z +$$
$$+ \frac{1}{2}\left\{\left[\tilde{V}_x\left(\frac{\partial\tilde{V}_x}{\partial x} - 2k_0\tilde{V}_y\right) + \tilde{V}_z\frac{\partial\tilde{V}_x}{\partial z}\right]\mathbf{e}_x + \left[\tilde{V}_x\frac{\partial\tilde{V}_z}{\partial x} + \tilde{V}_z\left(\frac{\partial V_z}{\partial z} - 2k_0\tilde{V}_y\right)\right]\mathbf{e}_z\right\}.$$

Hence, with an allowance for estimates (5.2) and (5.3), it is evident that, although the $\tilde{\mathbf{V}}e^{2ik_0y}$ component of the first-approximation solution produces in principle Reynolds stresses, which affect the velocity field $\mathbf{V}_{2c}$, these stresses are nevertheless small for $k_0 >> 1$ in comparison with the term $(\mathbf{V}_c\bullet\nabla)\mathbf{V}_c$ (their ratio amounts to $\sim 1/k_0^2$), and they can be neglected.

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NONSTATIONARY VORTEX FLOWS OF AN IDEAL INCOMPRESSIBLE FLUID

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UDC 532.5

As is known, analytic methods of sufficiently general nature were developed only for potential motions in the two-dimensional hydrodynamics of an ideal incompressible fluid while vortex flows were investigated for quite particular cases [1, 2]. Examples of unbounded plane flows with concentration vorticity that allow analytic description of unbounded plane flows with concentration vorticity that allow analytic description are certain systems of point vortices, vortex pairs, Karman street [1], a three vortex system [3], as well as a Kirchhoff vortex which is an elliptical domain of homogeneous vorticity ω rotating at the angular velocity $\Omega = \omega AB/(A + B)^2$ (A, B are the ellipse semiaxes). Goerstner [1] obtained a unique exact solution for vortex flows with a free boundary which describes trochoidal waves on the surface of an infinitely deep fluid [1].

Such a type of plane nonstationary biharmonically time-dependent vortex motions of a fluid is found in this paper as includes elliptical vortices and Goerstner waves as particular cases and, exactly as potential flows, allows the method of conformal transformation for the solution of specific problems. It is shown that in a certain sense the class of motions found is exceptional, viz., out of all possible solutions in Lagrange variables that contain a finite set of time frequencies, only the two-frequency solution obtained in this paper satisfies the hydrodynamics equations. However, this class describes only such vortex flows for which a reference system can be indicated where the trajectories of the fluid particles remain localized, which is not satisfied, say, for the shear layer.

The theory developed for these flows is used to investigate the self-consistent interaction of a nonstationary vortex domain with an external potential flow.

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